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The usage of stochastic matrices while learning the topic "Eigenvalues and eigenvectors of a matrix" in the course of Higher Mathematics

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Abstract. The article looks into teaching students mathematical modeling during the mastering of the topic "Eigenvalues and eigenvectors of a matrix" in the Linear Algebra. The modeling is based on the usage of matrices built based on Markov chains. The analysis of the researchers' papers shows that the usage of matrices of this type can be implemented while teaching students of different university specialties. The analysis of the prerequisites for the involvement of stochastic matrices in the consideration of students has helped to determine additional concepts and theorems that should be considered while learning "Eigenvalues and eigenvectors of a matrix". The theorem on the presence of stochastic vectors in a stochastic matrix can serve as a formula for systematizing problems that can present applications of eigenvalues and eigenvectors of the matrix in real-life situations. The results of the experiment conducted using the developed system of problems for students of different University specialties show the improvement of skills provided by mastering the topic "Eigenvalues and eigenvectors of a matrix".

1. Introduction

The study of Linear Algebra is provided in the course of Higher Mathematics at the university. Matrices, their determinants, solving a system of linear equations, and modeling using them are considered while learning the course. Several hours are also devoted to the consideration of eigenvalues and eigenvectors. However, the practical application of these concepts is almost not traced, and therefore the importance of learning them remains unclear to students. Mathematical modeling using eigenvalues and eigenvectors based on the usage of stochastic models is also neglected.

Many scientists are interested in mathematical modeling and its involvement in teaching Mathematics. Thus, Kavanagh and Galluzzo study mathematical modeling as an iterative process that consists of several components. They believe that one of the most important issues for students during modeling is time management, that is, during this process, students can

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easily get confused if teachers do not brainstorm with them and do not formulate the problem succinctly [1, 2]. In our opinion, indeed, mathematical modeling is a rather difficult process for students, but mathematical modeling itself will help motivate students to study Mathematics. One of the first models presented to students is a matrix, namely a stochastic matrix. In the course of mathematical modeling, a stochastic model is often built, which is a model of many processes in the surrounding world. Most stochastic processes are depicted in the form of Markov chains. Markov random processes, in particular Markov chains, as a mathematical model, are used to study processes in many branches of science, in particular in economics, ecology, sociology, technology, and industry [3–11].

Klumpenhouwer conducts research that presents an adapted mathematical framework for modeling transportation networks using Markov chains and presents a new tool that allows users to rapidly prototype and analyze stochastic networks using data or theoretical distributions [12].

Krenzler et al. develop algorithms for studying the modification of Jackson networks using Markov chains in a stochastic environment. The constructed algorithms respond to changes in the environment, the result of which is a clear calculation of the joint stationary distribution of the queue length vector and the environment [13]. Kenton views stochastic modeling as a form of financial modeling used to make investment decisions. Stochastic modeling is used in various industries around the world. The insurance industry, for example, relies heavily on stochastic modeling to predict what companies' balance sheets will look like at some point in the future. Some other sectors and fields depend on stochastic modeling, including stock investing, statistics, linguistics, biology, and quantum physics [14]. Brans studies Markov models as a stochastic variable system, and their use in business applications, forecasting, and machine learning [15]. Based on Markov chains, Zhang et al. model the process stochastically of diffusion and chemical reactions of chloride ions in concrete bridges, which will allow timely maintenance [16]. Bento considers the mathematical modeling of stochastic processes using a Markov chain. She describes the Markov model as a realistic tool for describing the world and creating long-term forecasts regarding some processes or systems [17].

From the listed studies, it can be seen that Markov chains have a fairly wide application in various spheres of life, so they require further research, which we consider based on eigenvalues and eigenvectors.

"Eigenvalues and eigenvectors of a matrix" is not a simple topic for students to understand. It is believed, that a properly and correctly composed system of problems on the topic will allow the student not only to practice the competencies, skills, and abilities to solve problems on the topic but also to demonstrate the application of the acquired knowledge in other branches of science and spheres of life, which will increase the motivation of students to study topics. So, having chosen the topic "Eigenvalues and eigenvectors of a matrix", it compiled a system of problems for students of various specialties. Systematized problems were aimed at developing certain skills. The separation of these skills can be demonstrated using the following Problem 1.

Problem 1. Two companies, X and Y, provide telephone, Internet, and digital television and share a market of 5 million people. Company X is not very customer-friendly and loses 12% of its customers to Company Y every year. Not all of Company Y's customers are satisfied with their service and every year 3% of Company Y's customers switch to Company X. We want to investigate how the number of customers of both companies will change from sometimes. It is presented in table 1 the solving to the problem and highlighted the skills that the student needs for this.

Table 1: Solving Problem 1.

Skills needed to solve the problem	Steps of problem solving
1. The student knows what discrete-time Markov chains are and how they are formed	This is a special case of a Markov chain.
2. The student knows how to enter notation and create a system of equations that reflects the process described in the problem	The number of customers of company X in year n is denoted by x_n and the number of customers of company Y in year n by y_n Then the sequences $\{x_n\}$ and $\{y_n\}$ satisfy the following system of equations: $x_{n+1} = 0.88x_n + 0.03y_n$, $y_{n+1} = 0.12x_n + 0.97y_n$.
3. The student is able to rewrite the system of linear equations in matrix form	The system of equations is rewritten in matrix form $\forall n \in N: \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.88 & 0.03 \\ 0.12 & 0.97 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$
4. The student knows how to recognize stochastic matrices, define them formally and knows which definitions and theorems of the topic "Eigenvalues and eigenvectors of a matrix" work with them	Based on the recursion, a mathematical model is obtained that shows the relationship between the number of customers in year n and the number of customers at the beginning (so-called initial conditions) $$\forall n \in N: \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0.88 & 0.03 \\ 0.12 & 0.97 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$ with stochastic matrix $$\begin{pmatrix} 0.88 & 0.03 \\ 0.12 & 0.97 \end{pmatrix}$$ This switching matrix has some special characteristics: all its elements are non-negative, and the sum of the elements in each column is 1. Each element of this matrix represents the probability that a customer will be loyal to his operator or switch from it to another operator.
5. The student knows how to find the eigenvalues of the matrix	The eigenvalues of the matrix are found $$\begin{pmatrix} 0.88 & 0.03 \\ 0.12 & 0.97 \end{pmatrix}$$

Skills needed to solve the problem	Steps of problem solving
6. The student knows the theorem on the presence of stochastic vectors of the stochastic matrix and knows how to work with it. *Theorem.* If one takes $M \in R^{nxn}$ as a stochastic matrix, where $m_{i,j} \geq 0$ for all $i, j \in \{1; \cdots ; n\}$. Then (a) there exists a unique stochastic vector v^* which is the corresponding eigenvector to the eigenvalue 1. (b) eigenvalues λ of the matrix M other than 1 satisfy $\|\lambda\| < 1$. (c) for all stochastic vectors v one has $\lim_{n\to infty} M^n v = v^*$.	which is a stochastic matrix with only positive elements, and has eigenvalues 0.85 and 1, the latter as expected, and the former indeed satisfies $0.85 < 1$. Since the transition matrix has only positive elements, the stochastic vector existence theorem can be applied. A unique stochastic eigenvector $(x^*;\ y^*)$, corresponding to the eigenvalue 1 is determined. Direct calculation leads to the equation $0,12x^* = 0,03y^*$, so the vector (1; 4) and its nonzero multiples are eigenvectors, corresponding to the eigenvalue 1. Of course, only the eigenvector $$(x^*;\ y^*) = \left(\frac{1}{5};\ \frac{4}{5}\right)$$ is stochastic. When solving the system of equations, it is mentioned that the initial values $(x_0;\ y_0)$ satisfy $x_0 + y_0 = 1$. The item (c) of the theorem applies, i.e.
7. The student has skills to analyse the results obtained as a result of solving the model and come to conclusions	One comes to the same conclusion as before: regardless of the initial market shares $(x_0;\ y_0)$, Company Y will have four times as many customers as Company X or, equivalently, Company X will have a market share of 20%, and company Y will have a market share of 80%.

After distinguishing the system of skills, it was expected to develop not only skills directly related to the mastery of the topic but also skills. This development will help to understand where the calculations of eigenvectors and eigenvalues of the matrix can be used.

So, the study aimed to develop a system of problems for learning the topic "Eigenvalues and eigenvectors of a matrix" to students of different university specialties. With the help of systematized problems, show students the usage of eigenvectors and eigenvalues in various branches of science and spheres of life. To conduct an experiment in which students of different university specialties will take part, to check the impact of the developed system of problems on the development of students' skills.

2. Method

The research method was the systematization of problems to show students the usage of eigenvectors and eigenvalues in various fields of science and spheres of life. Before developing the problems, several related theoretical propositions were identified, which became the basis for their systematization and solution.

1. Definition of stochastic matrix and stochastic vector.
   It is taken $M \in R^{nxn}$ and $\nu = (\nu_1; \cdots ; \nu_n) \in R^n$.
   If $m_{i,j} \geq 0\ \forall i, j \in \{1; \cdots ; n\}$ and $\sum_{i=1}^{n} m_{i,j} = 1\ \forall j \in \{1; \cdots ; n\}$ then the matrix M is called a stochastic matrix, which is sometimes also called a probability matrix or a Markov matrix.
   If $\nu_i \geq 0\ \forall i \in \{1; \cdots ; n\}$ and $\sum_{i=1}^{n} \nu_i = 1$, then the vector ν is called a stochastic vector, sometimes also called a probability vector.

2. The theorem on the product of a stochastic vector by a stochastic matrix. It is taken $M \in R^{nxn}$ as a stochastic matrix, and $(\nu_1; \cdots ; \nu_n) \in R^n$ – a stochastic vector. Then the transposed $M\nu^t$ is also a stochastic vector.
   Proof. With $m_{i,j} \geq 0 \; \forall i, j \in \{1; \cdots ; n\}$ and $\nu_i \geq 0 \; \forall i \in \{1; \cdots ; n\}$ the vector $M\nu^t$ has only non-negative elements. In addition,
   $$\sum_{i=1}^{n} \left(M\nu^t\right)_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} m_{ij}\nu_j\right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} m_{ij}\right)\nu_j = \sum_{j=1}^{n} \nu_j = 1.$$
   What had to be proved.
3. Definition of Markov chains
   A Markov chain is a sequence of stochastic vectors (written as column vectors) $(\nu_0; \nu_1 = M\nu_0; \cdots ; \nu_n = M^n\nu_0; \cdots)$ obtained by successive multiplication by the stochastic matrix M, starting from the stochastic vector ν_0.
4. The theorem on the presence of 1 among the eigenvalues of the stochastic matrix.
   It is taken $M \in R^{nxn}$ as a stochastic matrix. Then 1 is an eigenvalue of M.
   Proof. Consider the matrix $M \; - -E_n$. Since the sum of the elements of each column M is equal to 1, in the matrix $M \; - -E_n$ the sum of the elements of each column is equal to 0. As a result, if one sums all the rows of $M \; - -E_n$ it will be got a row with all elements equal to 0. This means that the rows of the matrix $M \; - -E_n$ are linearly dependent. Therefore, $det(M \; - -E_n) = 0$ and 1 is an eigenvalue of M. What had to be proved.
5. The theorem on the presence of stochastic vectors in a stochastic matrix. It is taken $M \in R^{nxn}$ as a stochastic matrix, with $m_{i,j} \geq 0 \; \forall i, j \in \{1; \cdots ; n\}$. Then
   (a) there exists a unique stochastic vector ν^* which is the corresponding eigenvector to the eigenvalue 1.
   (b) eigenvalues λ of the matrix M other than 1 satisfy $|\lambda| < 1$.
   (c) for all stochastic vectors ν we have $\lim_{n\to\infty} M^n\nu = \nu^*$.
   The last theorem has become the main one while systematizing and solving problems. It is considered the system of problems that will allow students of different university specialties to demonstrate the application of eigenvalues and eigenvectors of the matrix in practice (table 2). During the systematization of problems, the peculiarities of the chosen specialties were taken into account.

Table 2: Problems' system.

Code specialties	Problems
073 Management; 075 Marketing; 076 Entrepreneurship, Trade and Exchange Activity; 111 Mathematics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)	1,000 employees are constantly active at the big festival. To ensure that these employees remain motivated, they are regularly involved in other activities. There are three types of activities, namely security, sales, and cleaning. From past experience, the organizer knows exactly how many employees should perform each type of activity at any given time. Therefore, the organization uses the following table 2.1 every half hour to determine how employees switch to another activity, because this way the requirements are met in the best way

<table>
<tr><th>Code specialties</th><th>Problems</th></tr>
<tr><td></td><td>

Table 2.1: Employees' switching to another activity

<table>
<tr><th>to \ from</th><th>security</th><th>sales</th><th>cleaning</th></tr>
<tr><td>security</td><td>0.4</td><td>0.3</td><td>0.2</td></tr>
<tr><td>sales</td><td>0.1</td><td>0.4</td><td>0.1</td></tr>
<tr><td>cleaning</td><td>0.5</td><td>0.3</td><td>0.7</td></tr>
</table>

The numbers in the first column mean that every half hour the security workers will switch to another activity as follows: 4 out of 10 will continue to security, 1 out of 10 will switch to sales and 5 out of 10 will switch to cleaning. At the start of the festival, 500 employees will be engaged in security, 400 in sales, and 100 in cleaning.
(a) Estimate how many employees will be involved in security after the festival is over for more than one hour.
(b) Calculate how many workers will be removed in the long run. This number can be approximated by assuming that the festival lasts indefinitely
</td></tr>
<tr><td>091 Biology; 203 Horticulture and Viticulture; 206 Horticulture; 111 Mathematics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)</td><td>
The color of the dragon flower is a property determined by two genes, which we denote as A and a. A plant with genotype AA is red, a plant with genotype Aa is pink, and a plant with genotype aa is white. A plant will pass to its offspring if it has two genes; we assume that both genes have the same probability of transmission.
A florist has a large population of dragon flowers with a red colour (genotype AA), which are called generation 0 plants. The florist fertilizes each of these plants with pollen from plants with a pink colour (genotype Aa) from another population. Each plant has one new plant as an offspring, and we call these new plants generation 1 plants.
Since the pollen used for fertilization is probably one half containing the A gene and half is likely to be a gene, while the generation 1 plants have already received the A gene from the generation 0 plant, half of the generation 1 plants are red and the other half is pink.
The florist repeats this process: plants of a certain generation are fertilized with pollen from a plant with a pink colour so that each plant has a new plant as the offspring of the next generation. It is denoted by x_n, y_n and z_n, respectively, the fractions of plants with red, pink, and white colour in generation n. That is, $x_0 = 1, y_0 = 0, z_0 = 0$ and $x_1 = 0.5, y_1 = 0.5, z_1 = 0$.
(a) Calculate x_2, y_2 and z_2.
(b) Express x_{n+1}, y_{n+1} and z_{n+1} in terms of $x_n, y_n and z_n$ for $n \in N$.
(c) Compute the limit of the sequences xn, yn and zn. What is the meaning of these limit? [18]
</td></tr>
</table>

Code specialties	Problems
101 Ecology; 103 Earth Sciences; 201 Agronomy; 111 Mathematics; 113 Applied Mathematics; 014 Secondary Education (Mathematics	An ecosystem that contains 4 (sources) of atmospheric pollution is being studied: $\omega_1, \omega_2, \omega_3$ and ω_4. For one specific period of time, these points undergo the changes described below. In point ω_1 as a source of pollution, 0.1 of the initial pollution remains there, 0.18 of it goes to point ω_2, 0.2 to point ω_3, 0.3 to point ω_4, and 0.22 of all pollution is dispersed. Similarly, in the same period of time, 0.11 of the initial pollution remains in point ω_2, 0.29 of it moves to point ω_1, 0.15 to point ω_3, 0.1 to point ω_4, and 0.46 is dispersed. At point ω_3, the picture is like: 0.1 pollution remains here, 0.12 goes to ω_1, 0.32 to ω_2 and 0.2 to ω_4, and 0.26 is dispersed in the atmosphere. At point ω_4, 0.05 of pollution remains, 0.23 of it goes to ω_1, 0.19 to ω_2, 0.13 to ω_3, and 0.4 is dispersed in the atmosphere. Find the vector f when it is known that the value of t is large enough, and also the vector of pollution restrictions is given $q = (10; 10; 10; 10)$ [18].
51 Economics; 101 Ecology; 103 Earth Sciences; 201 Agronomy; 111 Mathematics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)	Statistical processing of observations of the meteorological service, carried out in the summer for a certain city of Ukraine, gave the following results: • if a certain day was warm and cloudless, then the probability that the same weather will remain the next day is equal to 0.6; the probability that it will change to windy weather is 0.25, and to rainy weather – 0.15; • when the weather was windy, the probability that it will remain the same the next day is 0.32, and the probability that it will change to quiet sunny weather is 0.46, and to rainy weather – 0.22; • and if the weather was rainy, the probability that it will not change the next day is 0.26, and the probability that it will change to windy or sunny calm weather is 0.29 and 0.45, respectively. Determine what the weather will be like in the summer of a certain city of Ukraine most often [18].
112 Statistics; 111 Mathematics; 113 Applied Mathematics; 011 Educational, Pedagogical Sciences; 014 Secondary Education (Mathematics)	According to the results of processing statistical information about the educational process of some University of Ukraine, the following data about its average student were obtained: • a 1st-year student with a probability of 0.1 stops his studies due to failure, with a probability of 0.25 remains a freshman for another year and with a probability of 0.65 transfers to the 2nd year; • a 2nd-year student with a probability of 0.15 drops out due to failure, with a probability of 0.3 repeats the 2nd year, with a probability of 0.55 goes to the 3rd year; • a 3rd-year student drops out with a probability of 0.22, becomes a sophomore with a probability of 0.31, transfers to the 4th year with a probability of 0.57; • a 4th-year student drops out with a probability of 0.12, becomes a second-year student with a probability of 0.2, and transfers to the 5th year with a probability of 0.68; • a 5th-year student drops out with a probability of 0.05, becomes a sophomore with a probability of 0.15, presents a thesis with a probability of 0.8 and leaves the University as a certified specialist.

<table>
<tr><th>Code specialties</th><th>Problems</th></tr>
<tr><td></td><td>Determine what percentage of students leave the University as a certified specialist [18].</td></tr>
<tr><td>141 Power Engineering, Electrical Engineering and Electromechanics; 112 Statistics; 242 Tourism; 111 Mathematics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)</td><td>Electricity consumption in summer is closely related to air temperature. Therefore, when planning the production and use of electricity for each day, the power company that supplies the city's population with electricity must take into account the probability of hot, moderate or cool weather. Long-term observations have shown that the probability that the weather will be hot, moderate or cool tomorrow depends only on the weather today – hot, moderate or cool (see table 2.2).

Table 2.2: Employees' switching to another activity

<table><tr><th></th><th>Hot</th><th>Moderate</th><th>Cool</th></tr><tr><td>Hot</td><td>60%</td><td>30%</td><td>20%</td></tr><tr><td>Moderate</td><td>30%</td><td>50%</td><td>40%</td></tr><tr><td>Cool</td><td>10%</td><td>20%</td><td>40%</td></tr></table>
(a) What will the weather be like on Thursday if it is moderate on Tuesday?
(b) What is the prevailing summer weather in this region and by what percentage? [18]</td></tr>
<tr><td>275 Transport Technologies; 112 Statistics; 242 Tourism; 111 Mathematics; 051 Economics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)</td><td>The purpose of the study was to find out what type of transport (trolleybus, bus, subway) the average resident of the capital uses to commute. It was found that when he/she went to work by bus on a certain day of the working week, the probability that he/she would also use the bus the next day was 0.6, and the probability that he/she would change the bus to a trolleybus or subway is 0, respectively .35 and 0.05. If he/she first travelled by trolleybus, the probability that he/she would not change the mode of transport the next day is 0.7, and the probability that he/she would change the mode of transport to the bus or subway is 0.25, and 0.05, respectively. Finally, if the resident first used the subway, the probability that the next day would be the same mode of transport is 0.8, and the probability of changing to a bus or trolleybus is 0.05 and 0.15, respectively.
Make a one-step transition matrix of a system that has three states corresponding to three types of urban transport used by a resident, and determine the probability that he/she would use the bus on Wednesday and Friday if on Monday he/she travelled by subway (a five-day work week). What type of transport will the average resident of the capital use to commute most often in the future [18]</td></tr>
<tr><td>111 Mathematics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)</td><td>In order not to be bored due to the measures against COVID-19, Alexander, Beth and Charles came up with a game. Each of them chooses an initial account. In each round of the game, each of them receives as the average of the other two players' scores. For example: If they choose starting scores of 3, 193 and 59 respectively, then the scores change as follows (see table 2.3).</td></tr>
</table>

<table>
<tr><th>Code specialties</th><th>Problems</th></tr>
<tr><td></td><td>
Table 2.3: Score after rounds
<table>
<tr><th></th><th>Alexander</th><th>Beth</th><th>Charles</th></tr>
<tr><td>Starting score</td><td>60%</td><td>30%</td><td>20%</td></tr>
<tr><td>Score after round 1</td><td>30%</td><td>50%</td><td>40%</td></tr>
<tr><td>The score after the 2nd round</td><td>10%</td><td>20%</td><td>40%</td></tr>
</table>
The winner is the player who has the most points after 999 rounds.
(a) Express the score of each of the three players after round n in terms of the points after round n – 1. (Here n is an arbitrary integer greater than or equal to 1, and we denote by ‘scores after round 0’ the initial scores).
(b) If Alexander and Beth choose their starting score and reveal it, what strategy do you recommend Charles use to win the game? Explain why.
(c) What happens to the scores if they do not stop after 999 rounds, i.e., the game goes on forever?
</td></tr>
<tr><td>075 Management; 111 Mathematics; 051 Economics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)</td><td>
80,000 visitors are constantly present at the concert. They are divided into three stages, and the organization measures movements every fifteen minutes between these stages in percentages, as in the following table 2.4.
Table 2.4: Movements every fifteen minutes between stages
<table>
<tr><th>from / to</th><th>stage A</th><th>stage B</th><th>stage C</th></tr>
<tr><td>stage A</td><td>50%</td><td>40%</td><td>30%</td></tr>
<tr><td>stage B</td><td>30%</td><td>30%</td><td>30%</td></tr>
<tr><td>stage C</td><td>20%</td><td>30%</td><td>40%</td></tr>
</table>
The percentages in the first column mean that every fifteen minutes the visitors to stage A are distributed as follows: 50% stay at stage A, 30% go to stage B, and 20% go to stage C.
At the start of the concert, there are 40,000 people at Stage A, 20,000 at Stage B and 20,000 at Stage C.
(a) How many visitors will there be at stage A after 30 minutes?
(b) How many visitors will there be after k 15 minutes at stage B?
(c) How many people are in attendance at the end of the concert at stage C?
</td></tr>
<tr><td>051 Economics; 292 International Economic Relations; 111 Mathematics; 113 Applied Mathematics; 014 Secondary Education (Mathematics)</td><td>
Economists study the economic condition of 150 countries and divide them into two groups: those in recession and those that are not in recession. If a country is in a recession, the probability that the country will still be in a recession next year is 70%. Countries that are not in recession have an 80% chance of still not being in recession in the next year. Currently, 50 out of 150 countries are in recession. How many countries do you think are in a long-term recession?
</td></tr>
</table>

3. Results

To check the effectiveness of using the system of problems in which stochastic models are used in classes of Linear Algebra, an experiment was conducted. The base for experimenting was Universities: National University of "Kyiv-Mohyla Academy" and Kryvyi Rih State Pedagogical University and students of majors: 073 Management; 075 Marketing; 076 Entrepreneurship, Trade and Exchange Activity; 113 Applied Mathematics; 014 Secondary Education (Mathematics).

The main tasks of the experiment were: the study of learning "Eigenvalues and eigenvectors of a matrix" by students of different majors; developing and implementing the system of problems that can demonstrate the usage of eigenvectors and eigenvalues of the matrix while constructing stochastic models; analysis of the experiment's results.

The control group (CG) and the experimental group (EG) were formed at the beginning of the experiment to determine the level of students' mathematical preparation after testing the first few topics of the Linear Algebra section in the Higher Mathematics course. After the end of the experiment, students were tested again. At the beginning and the end of the experiment, students of the control and experimental groups solved the same system of problems developed by the authors of this study.

The experiment lasted two weeks, during which the eigenvalues and eigenvectors of the matrix were studied in the course of the considered section of Higher Mathematics. 125 students took part in the experiment, of which 64 students were included in the control group and 61 students in the experimental group:

- the control group (CG) included students of majors 076 Entrepreneurship, Trade and Stock Exchange Activity; 113 Applied Mathematics; 014 Secondary education (Mathematics) from the National University of "Kyiv-Mohyla Academy" and Kryvyi Rih State Pedagogical University. Learning "Eigenvalues and eigenvectors of a matrix" in these groups was carried out with the usage of the standard problems;
- students who majored in 073 Management, 075 Marketing, 014 Secondary Education (Mathematics) from the National University of "Kyiv-Mochyla Academy" and Kryvyi Rih State Pedagogical University were involved in the experimental group. The experiment was carried out using the developed system of problems, in which modeling on stochastic models was considered using the eigenvalues and eigenvectors of the matrix.

At the beginning of the experiment, students in the control and experimental groups wrote the same test, dedicated to determinants, matrices, their properties, and solving systems of linear equations. The control was carried out to show the equivalence of the selected groups. The mode of test results for the experimental and control groups coincided. Then in the control group, the presentation of eigenvalues and vectors was carried out according to the traditional method, and in the experimental group, the developed system of problems was used.

The test, which was offered to students at the final stage of the experiment, contained three types of problems. The first type of problem is aimed at working out the steps of calculating the eigenvalues and eigenvectors of the matrix, namely, solving the equations given in the form of determinants and finding the fundamental system of solutions of an indeterminate homogeneous system of linear equations. This type of problem tests the skill to rewrite a system of linear equations in matrix form (Skill 1). The second type of problem is to find eigenvectors and eigenvalues of matrices of different order. These problems are aimed at testing students' skills to calculate eigenvectors and eigenvalues of a matrix (Skill 2). The third type of problem includes tasks on the application of eigenvectors and eigenvalues of the matrix, namely, finding the canonical equation of the second-order line, as well as tasks on developing a linear exchange model (international trade model). This type of problem was given to test the skill to build mathematical models for a problem, as well as the skill to adapt the obtained results following the condition of the problem and the ability to come to a specific conclusion (Skill 3).

It is presented in the table 3 and the diagram (figure 1) the percentage of correctly completed problems of the first, second, and third types, that is, the level of formation of the corresponding

Table 3. The percentage of formation of the relevant skill in CG and EG.

	Skill 1	Skill 2	Skill 3
Percentage in KG	84.4	67.2	54.7
Percentage in EG	91.8	83.6	78.7

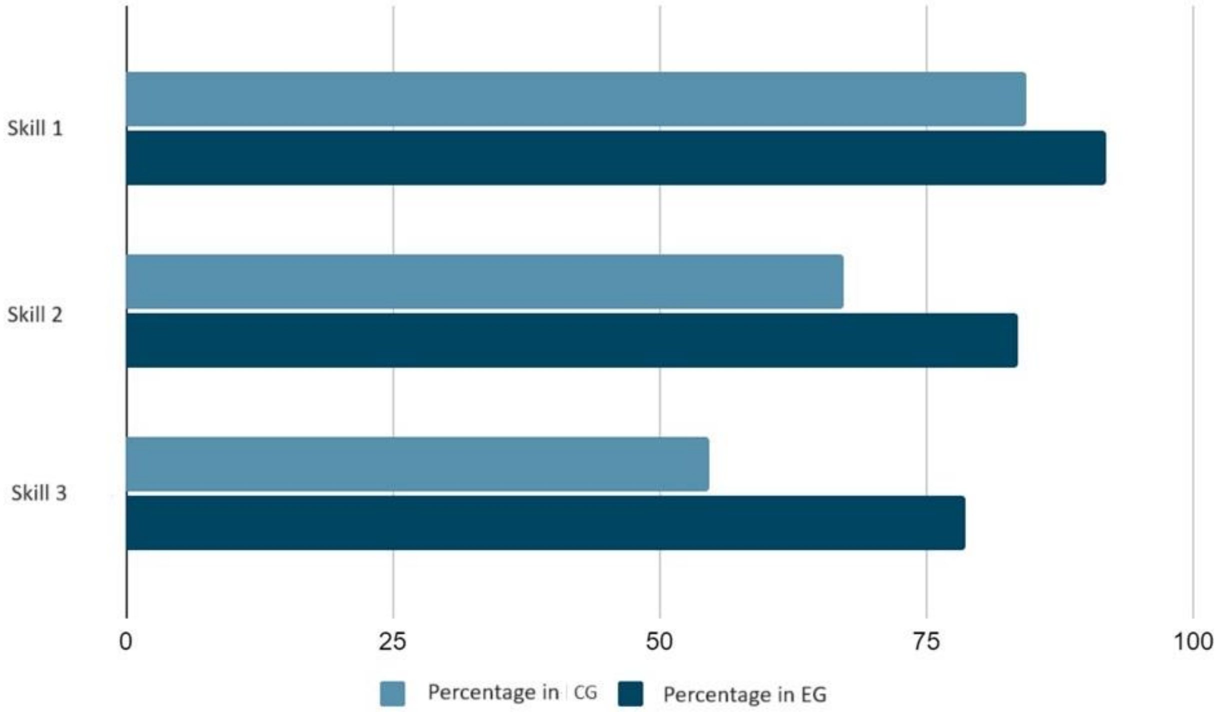

Figure 1. Results of the test at the final stage of the experiment

skill in each group.

It is clear from the table and diagram that the percentage of skills 1, 2, and 3 in the experimental group is higher than in the control group, for the first skill 7.4%, for the second skill – by 16.4%, and for the third skill – by 24%. It confirms the effectiveness of the problem system that was developed, in which modeling on stochastic models is considered using eigenvalues and eigenvectors of the matrix.

4. Discussion

One of the issues of linear algebra is the calculation of the eigenvector of a matrix associated with a known eigenvalue. This issue can be presented on realistic problems by replacing arbitrary matrices with stochastic ones and performing mathematical modeling on their basis. This approach, according to Vlasenko et al. [19], contributes to improving students' understanding of the importance of calculating the eigenvector and the eigenvalue of the matrix. This opinion was taken into account, so the standard matrices were replaced by matrices based on Markov chains.

Markov chains are widely used in various scientific fields. Their involvement can be found in both natural and social sciences. These matrices are used to explain the problems of movement in time or space, respectively, the transition from one state to another [20]. This type of matrix can be seen during the description, analysis, forecasting, and development of various features, such as migration, employment, and changes in urban systems and landscape [21–23]. So, we had a sufficient number of examples of their involvement to develop a system of problems that can be used while teaching the topic "Eigenvalues and eigenvectors of the matrix" to students given their major. Setting the conditions of the problems was carried out based on observing the stages of their solution to the conclusions of the theorem on the existence of stochastic vectors of the stochastic matrix $M \in R^{n\times n}$ where $m_{i,j} \geq 0 \; \forall i, j \in \{1; \cdots ; n\}$.

Therefore, the theorem became a formula for the systematization of problems with which the students of the experimental group worked. Of course, the students of the same group were also familiarized with all the theoretical provisions that ensure the modeling and solving obtained formalized models. The results of the experiment confirmed the expediency of using the developed problem system for learning the topic "Eigenvalues and eigenvectors of the matrix".

5. Conclusions

There is no doubt about the importance of teaching students mathematical modeling when learning Linear Algebra topics in the Higher Mathematics course. Learning the topic "Eigenvalues and eigenvectors of a matrix" is no exception. However, finding the eigenvalues and their corresponding eigenvectors of the matrix causes difficulties for students due to the lack of understanding of their use in real life. Such a misunderstanding can be eliminated through the usage of matrices built based on Markov chains.

Stochastic matrices have wide application in various scientific fields (data analysis, probability theory, statistics, mathematics, computer science, and population genetics) and life practice (migration, employment, changes in urban systems and landscape, etc.). Therefore, using matrices of this type can be demonstrated to students of different University specialties. Provided students consider stochastic matrices, which are also called probability matrices, Markov matrices, or substitution matrices, additional concepts and theorems should be presented to students while studying the topic "Eigenvalues and eigenvectors of a matrix". Among them: the definition of a stochastic matrix and a stochastic vector, the theorem on the product of a stochastic vector by a stochastic matrix, the definition of Markov chains, the theorem on the presence of 1 among the eigenvalues of a stochastic matrix, the theorem on the presence of stochastic vectors in a stochastic matrix.

The last theorem can be used as a basis for the systematization of problems that present applications of eigenvalues and eigenvectors of a matrix in real-life situations. This is explained by the fact that the stages of solving such problems must be carried out by observing the conditions and conclusions of this theorem. The usage of the developed system of problems and presentation of the theoretical foundations of its involvement while studying the topic "Eigenvalues and eigenvectors of the matrix" does not require a significant increase in hours. The results of the experiment show that students who work with the recommended system of problems demonstrate better results, which course to the improvement of skills ensuring mastery of the topic.

6. CRediT author statement

- *K. V. Vlasenko*: Conceptualization, Supervision, Project administration
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